

On the type of the temperature phase transition in $O(N)$ models within a perturbative analysis

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Abstract

We investigate the type of the temperature phase transition in the N component $\lambda\phi^4$ ($O(N)$) model of scalar fields. Actual calculations are carried out in the beyond-super-daisy approximation (BSDA). The cases $N = 1$ and larger N are considered separately. Using the solutions of gap equations we show that the character of the phase transition depends on the account for graphs BSDA. The role of different kinds of diagrams (especially the "sunset" one) is clarified. It is shown in a perturbation theory in the effective expansion parameter $N^{-1/3}$ that the kind of the phase transition depends on the value of coupling λ . It turns from a weak first-order to the second-order one for increasing λ . This is in agreement with the observation found recently for the $O(1)$ model in Monte Carlo simulations on a lattice. Comparison with results of other authors is given.

1 Introduction

Investigations of the temperature phase transition (PT) in the scalar N component $\lambda\phi^4$ theories, called $O(N)$ models, have a long history. Various approaches (perturbative, nonperturbative, analytic numeric as well as Monte Carlo (MC) simulations on a lattice) were used and different results have been obtained (see,

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for instance, Ref.[1] and recent papers [6], [7] for references). A final conclusion was not settled, yet. Nowadays, most calculations advocate a second order PT. However, a first order one is not excluded. In particular, using a partially resummed perturbative approach (resummation of two-particle irreducible (2PI) graphs) applied in the beyond-super-daisy approximation (BSDA), in either $O(1)$ or $O(N)$ models, the weak first-order PT was seen in [3]-[5]. Recently, in the $O(1)$ model it was shown in lattice MC calculations that the PT type depends on the value of coupling constant λ . For $\lambda \leq 10^{-3}$ a first order PT was detected and for larger λ it converts into a second order one [2]. This result is unexpected because it is usually assumed that the PT type is related with the analytic properties of some order parameter in the neighborhood of critical temperature T_c . It is also known that near the PT temperature the expansion parameter, which was initially small, is losing its smallness and perturbation theory in coupling constant becomes not reliable. Hence one can conclude that noted result requires further studies.

In recent papers [6], [7] the type of the temperature PT in both $O(1)$ and $O(N)$ models correspondingly was investigated with accounting for a "basketball" (or "sunset") diagram. It was stated that the PT becomes of second order just due to this diagram contribution. In Ref.[7] a detailed analysis was given by solving gap equations for particle masses within two approximations based on the 2PI resummations of perturbation series. One of them called "hybrid approximation" is an analytic one. These results and statements disagree with the ones in our earlier papers [3]-[5], where a weak first-order PT was detected. So, it is of interest to find out the origin of these discrepancies. This, in particular, is important because of essential differences in approaches applied in these investigations. It is also desirable to find possible causes for the dependence of the PT type on the value of λ since in Ref.[2] this behavior is the unexpected "fact of calculations".

In the present paper, on the base of Refs.[3]-[5], we analyze the corresponding gap equations, and solve them in different approximations for the BSDA graphs. One of them is the sunset diagram investigated in Refs.[6],[7]. In fact, this is only one term in a series of the BSDA bubble chain diagrams taken into consideration in Refs. [3] and [5]. In these papers neither the role of the sunset diagram nor reasons for changing of the PT type were investigated. Here, we consider in detail its influence both, the unique BSDA term and a particular element entering the complete series of diagrams. For simplicity of presentation, most calculations are carried out for the $O(1)$ model where the change in the type of the PT is detected in dependence on the kind of the BSDA diagrams taken into account. In particular, we confirm the result on the second order PT when the sunset diagram alone is accounted for. But this behavior switches to first-order type when a more complete series of diagrams, the bubble chains, is taken into consideration. Hence, the PT type is controlled by the analytic structure of the resummed diagrams independently of the value of λ . The case of $N \neq 1$ is discussed in short. We show that here the sunset diagram does not change the type of the PT which is of weak-

first-order at fixed N . We explain the causes of the discrepancies in the results obtained in Refs.[3],[4] and [6], [7]. A change in the type of PT in dependence on the coupling value was not observed. To investigate this possibility, we apply the perturbation theory in the effective expansion parameter $N^{-1/3}$ derived already in super daisy approximation (SDA) for large N [4]-[5]. Within this analysis we find a possible cause for this phenomenon.

The paper is organized as follows. In the next section necessary information on the calculations carried out in Refs.[3]-[5] is adduced. Then we consider in short the case of the $O(N)$ model and analyze the solutions of the gap equations in two cases, first, when the sunset diagram alone is accounted for and second, when other terms of the complete bubble chain series are taken into consideration. We show that in both cases the weak first-order PT happens. We also explain the origin of the discrepancies in the results obtained in Ref. [4] and Ref. [7]. In section 3 we consider the $O(1)$ model and show that in the case of one sunset diagram accounted for the second order PT takes place. However, it turns to the first order one when the complete series of bubble chain diagrams is taken into consideration. Section 4 is devoted to the investigation of the PT type as function of the value of λ in the limit $N \rightarrow \infty$ on the base of the perturbation theory in the effective expansion parameter $N^{-1/3}$. The dependence of the PT type is determined. The last section is devoted to conclusions, discussions and comparisons with other approaches.

2 PT in the BSDA for O(N) models

The thermodynamical properties of the model are described by the partition function

$$Z = \int D\varphi e^{-S[\varphi]}, \quad (1)$$

where φ is N component real scalar field, and the action is

$$S = \int dx \left(\frac{1}{2} \partial_\mu \varphi(x) \partial_\mu \varphi(x) - \frac{1}{2} m^2 \varphi(x)^2 + \frac{\lambda}{4N} (\varphi(x)^2)^2 \right). \quad (2)$$

Here, $\phi^2(x) = \sum_{i=1}^N \phi^i(x) \phi^i(x)$. More details on the model see in Refs.[3]-[5].

We use the second Legendre transform representation

$$W = S[0] + \frac{1}{2} Tr \log \beta - \frac{1}{2} Tr \Delta^{-1} \beta + W_2[\beta], \quad (3)$$

where $W_2[\beta]$ is the sum of all 1PI diagrams with propagators β . Further, $\Delta^{-1} = -\partial_\nu^2 + m^2$ is the inverse free propagator in Euclidian space and β is exact the propagator subject to the Schwinger-Dyson equation

$$\beta^{-1}(p) = \Delta^{-1} - \Sigma[\beta](p) \quad (4)$$

with $\Sigma[\beta](p) = 2\frac{\delta W_2}{\delta\beta(p)}$, which is the sum of all two-particle irreducible (2PI) self-energy graphs with propagators β . In the imaginary time formalism the operator "Tr" is

$$Tr_p = T \sum_{l=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \quad (5)$$

with the 4-momentum $p = (2\pi l/T, \vec{p})$. We indicate the functional argument β by square brackets and momentum by round ones.

Now let us remind our approximations. In Ref.[4], Eqs.(45), (55), the most general gap equations for Higgs, η , and Goldstone, ϕ , fields have been derived in the extremum of the free energy functional taken in the 2PI approximation which accounts for in the BSDA part an infinite series of bubble chain diagrams. We write down them here in the form

$$M_\eta^2 = 2\tilde{\Gamma} \left(m^2 - \frac{3\lambda}{N} \Sigma_\eta^{(0)} - \lambda \frac{N-1}{N} \Sigma_\phi^{(0)} + 2\Gamma \right) \quad (6)$$

for the Higgs field mass M_η and

$$M_\phi^2 = \frac{2\lambda}{N} (\Sigma_\phi^{(0)} - \Sigma_\eta^{(0)}) - gM_\eta^2 + f(\eta, \phi) \quad (7)$$

for the Goldstone field mass M_ϕ . The functions $\Gamma, \tilde{\Gamma}, \Sigma_\eta^{(0)}, \Sigma_\phi^{(0)}, f(\eta, \phi)$ are calculated in Eqs.(46), (50), (53) Ref.[4]. Eq.(6) at $N = 1$ coincides with the gap equation for the $O(1)$ model in Ref.[3]. These equations are derived from renormalized special series graphs entering 2PI free energy. The renormalization is fulfilled by the counter terms obtained at zero temperature. For more details see Refs.[8],[9].

Solving of the gap equations in the extremum of free energy is the key point of the approach used. Due to this choice, the physical masses of the finite temperature excitations are determined as the locations of the poles. Next what is important, as the consequence of this condition the term,

$$f(\eta, \phi) = \frac{\delta D}{\delta\eta(0)} - \frac{1}{N-1} \frac{\delta D}{\delta\phi(0)}, \quad (8)$$

with indefinite sign has appeared in Eq.(7). In the $O(N)$ models, this is the main reason for first-order PT (see below).

Now, let us write down the expressions in Eqs.(6)-(8) in the approximation when only the first terms (just the sunset contributions as in Ref.[6]) from the series of all diagrams derived in the Appendix of [4] are taken into account. We have (see Ref.[4])

$$\Gamma^{(1)} = -\frac{6\lambda}{N} Tr_q [\beta_\eta (\frac{a}{6} + \frac{\epsilon b}{2})] = \frac{\lambda^2}{N} (\frac{3}{N} S_\eta + \frac{N-1}{N} S_{\eta\phi_2}), \quad (9)$$

and

$$M_\phi^2 = \frac{2\lambda}{N}(\Sigma_\phi^{(0)} - \Sigma_\eta^{(0)}) - gM_\eta^2 + f(\eta, \phi) \quad (10)$$

for the Goldstone field mass

$$\frac{\delta D^{(1)}}{\delta \phi(0)} = \frac{\lambda^2}{3N}[(1 + 1/N)(N + 1 + 2\frac{N-2}{N+1})S_\phi + 3\frac{N-1}{N}S_{\phi\eta_2}], \quad (11)$$

where $\Gamma^{(1)}, D^{(1)}$ denote that only one term of a or b types is taken into consideration and we introduced the notations for the sunset diagrams

$$\begin{aligned} S_{\eta,\phi} &= \frac{T^2}{32\pi^2} \left(1 - 2\text{Log} \left[\frac{3M_{\eta,\phi}}{\mu} \right] \right), \\ S_{\eta,\phi_2} &= \frac{T^2}{32\pi^2} \left(1 - 2\text{Log} \left[\frac{M_\eta + 2M_\phi}{\mu} \right] \right), \\ S_{\phi,\eta_2} &= \frac{T^2}{32\pi^2} \left(1 - 2\text{Log} \left[\frac{2M_\eta + M_\phi}{\mu} \right] \right). \end{aligned} \quad (12)$$

Here, μ is a normalization scale. Calculation of these contributions by different methods see, for example, in Refs. [11]-[13]. In Eq.(10) the notation $\Sigma^{(0)} = \text{Tr}_q \beta(q)$ is used, where β is determined below in Eq.(13).

Obviously, for small M_ϕ , in Eq.(8) in the difference the signs of the leading log-terms $\text{Log}[\frac{3M_\phi}{\mu}]$ are dominant. This is because, as we showed in the SDA [4], the mass M_η is always larger than M_ϕ . Just these M_ϕ -terms control the type of the phase transition at high temperature when M_ϕ is small.

As we see from Eq.(9), there are no $\text{Log}[\frac{3M_\phi}{\mu}]$ terms in $\Gamma^{(1)}$. In this limit, in Eq.(11) the first term S_ϕ is dominant. Also, we see that $\text{Log}[\frac{3M_\phi}{\mu}]$ enters the function $f(\eta, \phi)$ in Eq.(8) with the sign "plus". So, it gives a negative contribution in the gap equation (10). This behavior is sufficient to generate the first order PT. Really, due to the negative sign noted there are two crossing of the parabola in the l.h.s. with the behavior in the r.h.s. One crossing is at small M_ϕ and other one at larger M_ϕ . As a result, substituting these values into Eq.(6) we have to obtain two solutions for M_η . This is because in the Γ the term $\text{Log}[\frac{3M_\eta}{\mu}]$ enters with the negative sign. So that at small M_η this contribution is positive and only one crossing of the l.h.s. with the r.h.s. is possible. In this way the first order phase transition is realized. Note that this is just due to the properties of the sunset diagrams, in contrast to what is stated in Ref.[6]. These authors erroneously related the first order type of the PT observed in Ref.[4] with the ansatz for the propagators

$$\beta_{\eta,\phi}^{-1}(p) = p^2 + M_{\eta,\phi}^2 \quad (13)$$

used therein. However, this is not the case and actual cause is the solution of the gap equations in the extremum of free energy. Note also that the behavior of the $\tilde{\Gamma}$ in Eq.(6) is not important, it does not lead to new crossings.

We have to note once again that the term with indefinite sign in Eq.(8) is the consequence of passing to the extremum of free energy. In contrast, in Ref.[7] the gap equations (Eqs.(6), (7) or (38), (39)) were solved out of the extremum of the free energy. So, the sunset contributions S_η and S_ϕ into the effective potential (Eq.(14), [7]) enter with the same signs. This is the reason why the second order PT was found. More details on this point are given in the last section.

As general conclusion of the above analysis we note that in the analytic calculations accounting for the sunset diagram alone the weak-first-order PT follows if the gap equations (6), (7) are solved in the extremum of free energy, that is if the condition $\partial W/\partial v^2 = 0$, where W is free energy and v is vacuum value of the Higgs field, is used explicitly.

3 Phase transition in $O(1)$ model

To investigate the $O(1)$ model we set $N = 1$ in Eq.(6) that omits the contribution of the ϕ fields. The structure of the functions Γ and $\tilde{\Gamma}$ was presented in Ref.[3].

The notations are:

$$\Gamma = Tr_q \beta(q) \Sigma_1(q) \frac{1 - 6\lambda \Sigma_1(q)}{1 + 3\lambda \Sigma_1(q)}, \quad (14)$$

(Eq.(18), [3]) and

$$\tilde{\Gamma} = 1 - 3 \frac{\delta \Gamma}{\delta \beta(0)}, \quad (15)$$

(Eq.(25), [3]), where β is given by Eq.(13) and

$$\Sigma_1(p) = Tr_q \beta(q) \beta(q + p), \quad (16)$$

(Eq.(20), [3]).

In order to be in complete correspondence with Ref.[3] we rewrite Eq.(6) in the form

$$M^2 = 2\tilde{\Gamma} \left(m^2 - \frac{3\lambda}{N} \Sigma^{(0)} + 6\lambda^2 \Gamma \right), \quad (17)$$

where $\Sigma^{(0)} = Tr_q \beta(q)$ and m is the zero temperature mass. This equation is also written in the extremum of free energy.

Now, we consider the approximation when among the BSDA terms the sunset diagram only is taken into consideration. In Eqs. (14), (15) this means that we have to put $\lambda = 0$. As a result we have

$$\Gamma^{ss} = Tr_q [\beta(q) \Sigma_1(q)] = S_\eta, \quad \tilde{\Gamma}^{ss} = 1, \quad (18)$$

where S_η is given in Eq.(12).

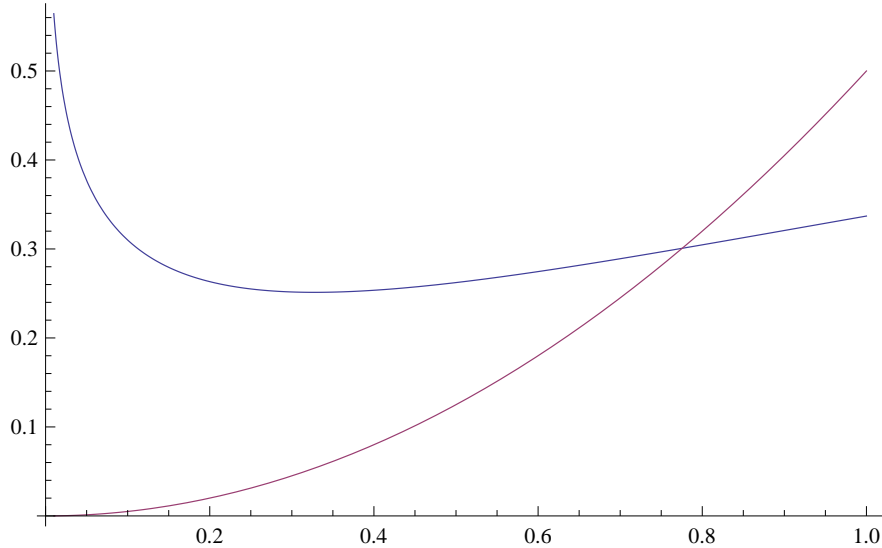


Figure 1: Solution of gap equation for $\lambda = 0.915$ and $\tau = 2$ with the sunset diagram taken into consideration, notations are given in the text.

We solve Eq.(17) graphically by presenting the l.h.s. and the r.h.s. in one plot. As the expression for $\Sigma^{(0)}(M)$ the high temperature expansion adduced in Eq.(55) of [3] is used:

$$\Sigma^{(0)}(M) = \frac{T^2}{12} - \frac{MT}{4\pi} + \frac{1}{16\pi^2} \left(M^2 \left(\ln \frac{8\pi^2 T^2}{m^2} - 2\gamma \right) + 2m^2 \right) + O(M^2). \quad (19)$$

Here, γ is Euler's gamma. To show graphical solutions we introduce the dimensionless variables $\tau = \frac{T}{m}$ and $\rho = \frac{M}{m}$ and set the normalization scale $\mu = m$ in S_η , although this is not essential.

In Figs.1-2 we show the results for two temperatures. Along the OX axis, the values of the scalar field mass ρ are measured. Along the OY axis, the values of the l.h.s. and the r.h.s. are present. As we see, there is only one crossing of the curves at some point $\rho = \rho(\tau)$. With increasing of the temperature the crossing point is shifted to smaller masses. This means the second order PT and is in agreement with the result in Ref.[6].

However, the sunset diagram is only one term in the bubble chain series resulting in Γ , Eq.(14). Since the numerator in this expression is not a function with definite sign, the value of Γ may change in dependence on the behavior of Σ_1 . In Ref.[3], Eq.(57), the high temperature expansion for this function is adduced:

$$\Sigma_1(0) = \frac{T}{4\pi M} + O(1). \quad (20)$$

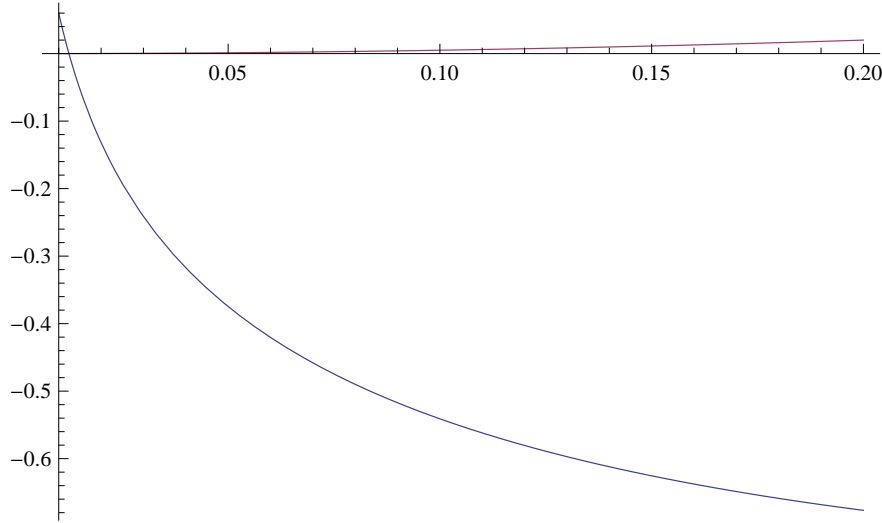


Figure 2: Solution of gap equation for $\lambda = 0.915$ and $\tau = 3$ with the sunset diagram taken into consideration, notations are given in the text.

Next, let us investigate the gap equation (17) with accounting for the full expression for Γ , Eq.(14), with this Σ_1 , as well as the complete $\tilde{\Gamma}$, Eq.(15). Corresponding solutions are shown in Figs.3-4. For this case, there are two crossings of the curves - one at small values of $\rho = \rho(\tau)$ and another at larger ones. This is typical behavior signalling a first order PT. Crossing at small ρ corresponds to the maximum of free energy at a given temperature. Crossing at larger $\rho(\tau)$ determines the value of the scalar particle mass in the minimum of the effective potential, that is, the physical mass of the excitation. Note here that the behavior of $\tilde{\Gamma}$ is not very essential. It influences a little bit the parameters of the PT. The type of the PT is completely determined by the function Γ . As a result, the first order PT is observed. Hence it follows that not only the sunset diagram but also all the other terms of the series are important and control the type of the PT. As concerns the dependence of the PT type on the value of λ , we do not see that in the given approximation for a wide interval of λ . Typical behavior is depicted in the above plots. Changing of the PT type follows in dependence on the expression for Γ , only. Note also that the choice of the truncated functions can influence the type of the PT. For example, in Ref.[3] more rough approximations for Γ were used. As a result, weak-first-order PT followed and a second order one was not observed.

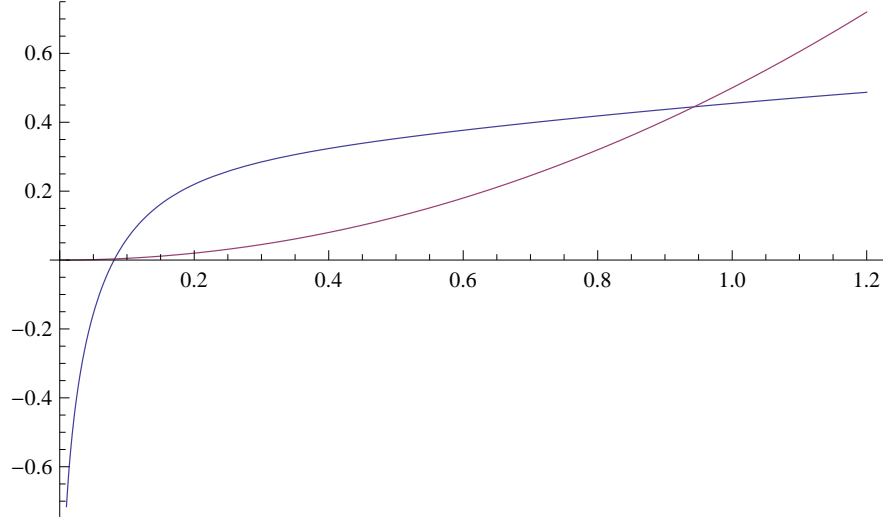


Figure 3: Solution of gap equation for $\lambda = 1$ and $\tau = 1.8$ with the complete set of bubble chain diagrams included, notations are given in the text.

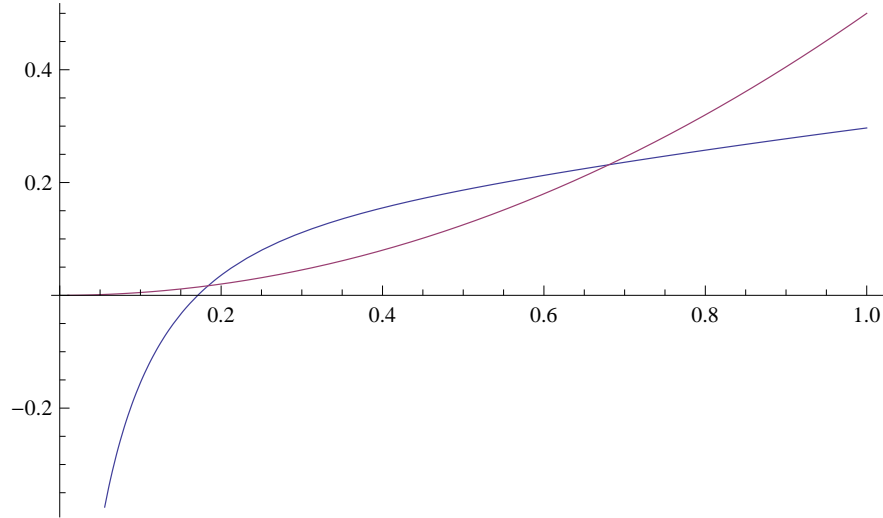


Figure 4: Solution of gap equation for $\lambda = 1$ and $\tau = 2$ with the complete set of bubble chain diagrams included, notations are given in the text.

4 λ -dependence in the $\frac{1}{N^{1/3}}$ expansion

In this section we investigate the λ dependence of the PT type within the perturbation theory based on the effective expansion parameter $N^{-1/3}$ near the PT temperature [4], [5].

First, let us adduce information necessary for what follows. This perturbation theory accounts for the contributions of the BSDA graphs or series of graphs. It is based on the particle masses $M_\eta^{(0)}$ and $M_\phi^{(0)}$ obtained as solutions to the gap equations which take into consideration the SDA graphs, only. That is, we have to omit the last terms in Eqs.(6), (7) and obtain (see Eq.(17) in [9]):

$$\begin{aligned} M_\eta^{(0)} &= \frac{\lambda T_+}{4\pi} \left(\frac{1}{(2N)^{1/3}} - \frac{1}{2N} + \cdots \right), \\ M_\phi^{(0)} &= \frac{\lambda T_+}{2\pi} \left(\frac{1}{(2N)^{2/3}} - \frac{1}{2N} + \cdots \right). \end{aligned} \quad (21)$$

These expressions were calculated in the limit $N \rightarrow \infty$, T_+ is the upper spinodal temperature. The choice of T_+ is motivated by simplicity of analytic expressions for this case.

The main steps of calculations are as follow. Each BSDA diagram containing n vertexes comes with the factor $(\frac{\lambda}{N})^n$ and has to be written with the masses (21). Then in the imaginary time formalism one must shift the three momentum of a loop $\vec{p} \rightarrow M\vec{p}$ and the temperature $T \rightarrow \frac{T}{M} \sim TN^{2/3}$, where for definiteness we have substituted the mass $M_\phi^{(0)}$. After these shifts the common N -dependent factor of the diagram is obtained. Hence, we can see that the high temperature approximation is well applicable for large N . As a result, the contribution of the static mode $l = 0$ in the Matsubara sum is dominant. As it is known, in the static limit ($l = 0$) a theory becomes effectively three dimensional. So, this approximation is sufficient for solving most problems of interest. This, in particular, means that we can calculate corrections to the SDA results in the three space dimensional $O(N)$ model.

In Ref.[9], as application, an infinite series of the bubble chain diagrams with the $M_\phi^{(0)}$ mass was calculated at the T_+ temperature. Remind, in the SDA approximation a weak-first-order PT was determined [4]. In this section, to investigate the λ -dependence of the PT, we recalculate these corrections in the $d = 3$ $O(N)$ model and check how the value of λ influences the difference between the lower T_- and the upper T_+ spinodal temperatures.

To realize that we use the result of summing up the series of the bubble chain diagrams in the restored phase (Eq.(26) in [9]):

$$\begin{aligned} D_\phi &= -\frac{\lambda^1}{N^2} T r_p \log \left(1 + \Sigma_\phi^{(1)}(p) N^{2/3} \right) \\ &+ \frac{\lambda^1}{N^{4/3}} T r_p \Sigma_\phi^{(1)}(p) - \frac{\lambda^1}{4N^{2/3}} T r_p \left(\Sigma_\phi^{(1)}(p) \right)^2. \end{aligned} \quad (22)$$

Here, $\Sigma_\phi^{(1)}(p)$ is defined in Eq.(16) and $Tr_p = \Sigma_{l=-\infty}^{l=+\infty} \int_{-\infty}^{+\infty} d^3p$. For simplicity, we restrict ourselves to the large N case and take into consideration the last term, only. Other terms give next-to-leading corrections.

Now, we have to renormalize this contribution. In Ref.[9] a general renormalization procedure was developed. In particular, it was shown for the vertexes and bubble chain graphs that the counter terms removing divergencies at zero temperature are sufficient for doing that at finite temperature, as it should be.

Here, we consider the renormalization of the last term in Eq.(22) in the three space dimensions. We present the term of interest in the equivalent form

$$D_\phi^{(3)} = -\frac{\lambda^1}{N^{2/3}} \frac{T}{(2\pi)^3} \int d^3p \beta(p, M_\phi) S_\phi(p, M_\phi), \quad (23)$$

where

$$S_\phi(\vec{p}^2, M_\phi) = \frac{T^2}{(2\pi)^6} \int d^3k d^3q \beta(k, M_\phi) \beta(q - k, M_\phi) \beta(q - p, M_\phi) \quad (24)$$

is the sunset diagram, and we set $l = 0$ in the Matsubara sums. This representation is convenient because we need it in the functional derivative $\delta D_\phi / \delta \beta$ as in Eq.(8). The function (24) was analytically calculated in Minkowski space-time for arbitrary d in terms of Hypergeometric functions [10]. It is not difficult to transform that expression for our case. More early calculations of the sunset diagram see in Refs.[11]-[13]. So, the counter term for this diagram C_3 can be easily found. As we see from the above expression, the integral is logarithmical divergent and so the counter term is a simple pole in the expansion over the deviation $2\epsilon = 3 - d$ in the limit $\epsilon \rightarrow 0$. It has to be subtracted from the $S_\phi(\vec{p}^2, M_\phi)$ to get a finite part. To obtain the renormalized $D_\phi^{(3)}$ we have to substitute the renormalized expression (24) in Eq.(23) and introduce a counter term which cancels the divergent part coming from the p-integration. This is standard procedure. In fact, the last step is not necessary for what follows because here we need in the contribution of the sunset diagram taken at $p = 0$.

Then we proceed as in Ref.[9] and solve the gap equations (6), (7) perturbatively. With the only term $D_\phi^{(3)}$ taken into account we have

$$M_\eta^2 = m^2 - \frac{3\lambda}{N} \Sigma_\eta^{(0)} - \lambda \frac{N-1}{N} \Sigma_\phi^{(0)} \quad (25)$$

and

$$M_\phi^2 = \frac{2\lambda}{N} (\Sigma_\phi^{(0)} - \Sigma_\eta^{(0)}) + f(M_\phi), \quad (26)$$

where now in the large N limit

$$f(M_\phi) = -\frac{2}{N-1} \frac{\delta D_\phi^{(3)}}{\delta \beta_\phi(0)} = \frac{\lambda}{N^{5/3}} \frac{T^2}{32\pi^2} \left(1 - 2 \log \frac{3M_\phi}{\mu} \right). \quad (27)$$

For the contributions $\Sigma_\eta^{(0)}$ and $\Sigma_\phi^{(0)}$ we use the leading terms in the high temperature expansion (19)

$$\Sigma^{(0)}(M_{\eta,\phi}) = \frac{T^2}{12} - \frac{M_{\eta,\phi}T}{4\pi} + \dots \quad (28)$$

To obtain perturbative solutions we write

$$\begin{aligned} M_\eta &= M_\eta^{(0)} + x, \\ M_\phi &= M_\phi^{(0)} + y, \end{aligned} \quad (29)$$

where the masses (21) are substituted and x, y are small corrections which must be calculated. The result is as follows [9]:

$$\begin{aligned} x &= \frac{1}{3} \frac{T_+}{32\pi} \frac{1}{N^{2/3}} \left(1 - 2 \log \frac{3M_\phi^{(0)}}{\mu} \right), \\ y &= \frac{1}{2} \frac{2^{2/3}}{3^{1/2}} \frac{T_+}{32\pi} \frac{1}{N} \left(1 - 2 \log \frac{3M_\phi^{(0)}}{\mu} \right). \end{aligned} \quad (30)$$

The values and signs of corrections depend on the choice of normalization parameter μ . This situation is known in field theory and particle physics. In the context of temperature PT in the standard model the choice of μ is discussed in Refs.[13],[14]. In Appendix B of the former paper some details of renormalization in $d = 3$ space dimensions relevant to our case are considered. This choice is also discussed in Ref.[7]. Although the result has to be independent of μ , in practice, in perturbation calculations μ is chosen to be of the order of typical mass in the problem. This removes large logarithmic corrections. In our case, we set $\mu = M_\phi^{(0)}$ because this is particle mass (21) at close to the T_c temperatures. With this choice, we see that the function (27) and corrections (30) have negative signs that decreases the masses of particles. Note also that in section 3 we set $\mu = m$. This choice was inessential because we were interested in qualitative behavior which is determined by the number of crossing of the curves presenting l.h.s. and r.h.s. of equations.

Similarly, the mass correction can be calculated in the restored phase where $M_\phi = M_\eta = M_r$. In Ref.[9] it is shown that if we present the mass as $M_r = M_r^{(0)} + z$, the correction z is given by the expression (see Eq.(52)in [9]):

$$z = \frac{f(M_r^{(0)})}{2M_r^{(0)} + \frac{\lambda T}{4\pi}}. \quad (31)$$

This is negatively valued function. As a result, the mass in the restored phase is also decreased. The mass $M_r^{(0)}$ was calculated already in the SDA in Refs.[4], [5]. It looks as follows:

$$M_r^{(0)} = -\frac{\lambda(N+2)T}{8\pi N} + \sqrt{\left(\frac{\lambda(N+2)T}{8\pi N}\right)^2 - m^2 + \frac{\lambda(2+N)T^2}{12N}}. \quad (32)$$

Clearly that the mass should be positive parameter.

In SDA, the temperature $T_-^{(0)}$ is determined from the condition $M_r^{(0)} = 0$. Its value is easily calculated with the expression (32):

$$T_-^{(0)} = \frac{2m}{\sqrt{\lambda}} \frac{\sqrt{3}\sqrt{N}}{\sqrt{2+N}}. \quad (33)$$

Corresponding value for $M_r = 0$ can be found from similar equation $M_r = M_r^{(0)} + z = 0$. To estimate the influence of the z correction, we consider numerical examples. Let us introduce dimensionless variables: $\tau = \frac{T}{m}$, $m_r^{(0)} = \frac{M_r^{(0)}}{m}$ and calculate $\tau_-^{(0)}$ and τ_-^z for $\lambda = 1, 5$. For definiteness we also take $N = 8$. We obtain for $\lambda = 1$, $\tau_-^{(0)} = 12$ and $\tau_-^z = 12.208$. For $\lambda = 5$ these numbers are $\tau_-^{(0)} = 2.4$ and $\tau_-^z = 2.408$. Hence, it follows that z -correction slightly increases the lower spinodal temperature. The same, of course, takes place for other values of the coupling and number of components N .

Now consider the temperature T_+ . In the SDA it was found that the relation holds [4], [5],[9]:

$$T_+^{(0)} = T_-^{(0)} \left(1 + \frac{9\lambda}{16\pi^2} \frac{1}{(2N)^{2/3}} \right). \quad (34)$$

As we see, for a given N the difference between these temperatures is increasing function of λ . So, in this approximation, the larger is λ the stronger is first order PT.

The BSDA correction to the $T_+^{(0)}$ was also calculated. The result is as follows (Eq.(61) in [9]),

$$T_+ = T_+^{(0)} \left(1 + \frac{9\lambda}{32\pi^2} \frac{(1 - 2\log \frac{3M_\phi^{(0)}}{\mu})}{N^{5/3}} \right). \quad (35)$$

Here, $T_+^{(0)}$ is given in Eqs.(33),(34) and for μ we have to substitute the value $M_\phi^{(0)}$ as before. Since the value $1 - 2\log 3 = -1.1972$ is negative, we observe a nontrivial behavior of the r.h.s. as a function of λ . It follows from the product of two brackets standing in Eqs.(34) and (35). The former one is increasing and the latter - decreasing function of λ . For a fixed N and sufficiently large λ , the brackets in Eq.(35) becomes enough small that turns T_+ to go down. As a result, the difference between T_+ and T_- is decreased. The weak-first-order PT derived in the SDA presents a tendency for converting into the second order one. Just such type behavior was observed for the $O(1)$ model on a lattice [2]. From Eqs.(34) and (35) it also follows that the small parameter has to be $\frac{\lambda}{16\pi^2} \frac{1}{(2N)^{2/3}}$ and therefore the value of λ can be sufficiently large and dependent on N . For example, for $N = 2$, the parameter is small for λ values 10-20.

Thus, close to the temperature T_c , the perturbation theory in the effective parameter $N^{-1/3}$ gives a possibility for determining causes for changing the PT

type dependently on the coupling value. It consists in specific λ -dependence of the sunset diagram contributions to T_+ . Since in three dimensions this is the only BSDA diagram with such type dependence, the importance of this contribution is clarified.

5 Conclusion

We have analyzed the temperature PT in the $O(1)$ and $O(N)$ scalar field models. Actual calculations were carried out in the BSDA for 2PI free energy functional. The gap equations were solved graphically within a high temperature approximation for Green's functions.

In the $O(1)$ model, we observed the change of the PT type in dependence on the BSDA diagrams taken into consideration. More definitely, for the case of the sunset diagram alone we found a second order PT, that is in agreement with the results of Ref.[6]. However, for the complete bubble chain series, containing the sunset one as a particular element, a first order PT was detected. We showed that the change of the PT type is related with the structure of the Γ term, Eq.(14), where the sunset diagram stands as a common factor. So, the properties of this diagram were taken into account completely. We can conclude that the sunset diagram does not uniquely control the type of the PT, other contributions are also important.

In Refs.[6], [7] it is, in particular, stated that the first order PT detected in Ref.[3] is the consequence of the ansatz for the full propagator β Eq.(13). However, as we showed above, this is not the case and the actual cause is a rough approximation for the Γ function Eq.(14) used in Ref.[3]. It is worth mentioning that, as it is demonstrated in Ref.[4], the SDA possesses a lot of attractive properties. One of them is a possibility to derive an effective expansion parameter $\sim N^{-1/3}$ near the PT temperature for large N . This allows to construct perturbation theory in this parameter. We have shown in the BSDA that in the $O(N)$ models the temperature PT is a weak-first-order. This also disagrees with the results obtained in Ref.[7], where the sunset diagram only was taken into account and a second order PT was observed. In section 2, we have demonstrated that in this case the first order PT follows from solving of the gap equations in the extremum of free energy. In fact, this is the main reason for the discrepancies in the obtained results. As we mentioned in the Introduction, solving of gap equations in the extremum of free energy is the key element of the approach applied in Refs.[3]-[5]. This requirement is based on the general principle of thermodynamics stated that physical masses have to be obtained in the minimum of free energy. In Eqs.(6),(7) this requirement is satisfied by construction.

Let us continue our comparison with the approach applied in Refs.[6], [7]. Two points of it may result in the discrepancies. First is the renormalization procedures used. In our consideration, a standard renormalization with zero tem-

perature counter terms is applied, as it is presented in the previous section and in Refs.[3],[9]. Moreover, for large N the static mode contribution is well applicable and the theory is effectively three dimensional one. In contrast, in Refs.[6], [7] renormalization is fulfilled at finite temperature with temperature dependent counter terms. As a consequence, uncontrolled finite temperature contributions affect the results. Second, in our method the solution to gap equations in the minimum of free energy results in the pole physical masses of particles. Just due to this choice the term (8) with indefinite appears and generates first-order PT. In Refs.[6], [7] the particle masses are calculated in two steps. At first, the gap equations are solved at arbitrary vacuum condensate giving the pole masses which are then substituted in the effective potential. Next by minimization of the potential and calculating its second derivative in the minimum the "curvature masses" are calculated. They are considered as the physical masses. In the course of these calculations, the sunset diagrams enter the stationary equation with the same signs. This is formal mathematical cause of the second order PT. We also remind that the solution of gap equations corresponds to the summation of the infinite series of loop diagrams that is a nonperturbative result. So, one joins these results with the minimization of the two-loop effective potential. This procedure does not coincide with solving of gap equations in the minimum of free energy. So, the results could be different. Besides, the usage of the two sorts of masses makes this calculation procedure complicated. We believe that pole masses are close related to the particle spectrum and should be found in the minimum of the effective potential not vice versa.

As concerns the dependence of the PT type on the coupling value observed in MC simulations on the lattice [2], it was not detected in analytic calculations within the solutions to gap equations fulfilled in the BSDA. In this approach, the PT type is determined by the number of crossings of the curves depicted in plots Figs. 1-4. This number is strictly related with the signs of the log-terms entering the dominant sunset diagram contributions. If this sign is positive at small M , the only one crossing happens and the second order PT follows. Otherwise the first order PT takes place. These properties are independent of the λ value. However, different kind resummations can be carried out that makes this sign out of any control in general.

It is also interesting to compare our results with the ones derived in analytic solutions of exact renormalization group equations [15]. In this approach, in the framework of the effective average action, the analytic solutions for the scale dependence of the potential have been obtained for $O(N)$ models in different space dimensions. Actual calculations were carried out in the limit $N \rightarrow \infty$. For $d = 4$, a second order PT has been found. This is in agreement with our results for large N . Here, it is interesting to note that for some relations between the parameters of the models a first order PT was observed. This behavior is similar to the PT λ -dependence observed in Ref.[2] and in section 4. Note also that detailed analysis and comparison for a number of other papers are given in

Refs. [3]-[5].

We have investigated the dependence of the PT type on the value of coupling in the perturbation theory based on the effective expansion parameter $N^{-1/3}$ near the PT temperature. We found by calculating the contribution of the bubble chain graphs D_ϕ (22) that this change is possible and this is the consequence of the sunset diagram contribution near the T_c . It includes the λ factor with the sign which acts to influence the PT type. Hence, we have to conclude that the problem on the PT type must be considered with accounting for this value. This point can be investigated with the MC methods on a lattice as in [2] for O(1) model. We left this problem for the future.

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